Int. J. Heat Mass Transfer. Vol. 36, No. 3, pp. 593–597, 1993
Printed in Great Britain

0017-9310/93 $6.00+0.00
Pergamon Press Ltd

Convective wall plume in power-law fluids

IOAN POP
Faculty of Mathematics, University of Cluj, R-3400 Cluj, CP 253, Romania

JIN KOOK LEE
Engineering Technology, Aircraft Accessories, Argo-Tech Corporation, Cleveland, OH 44117, U.S.A.

and

RAMA SUBBA REDDY GORLA
Department of Mechanical Engineering, Cleveland State University, Cleveland, OH 44115, U.S.A.

(*Received* 14 *May* 1991 *and in final form* 10 *April* 1992)

Abstract—This paper considers the steady-state free convection flow arising from a line thermal source positioned at the leading edge of a vertical adiabatic surface embedded in polymeric fluids. The wall plume depends on two parameters: the index of the power-law fluid (n) and the generalized Prandtl number (Pr). Precise conditions of finding similarity solutions for this problem are derived. A family of numerical solutions for n ranging from 0.2 to 2.0 and for $Pr = 10$ and 100 is reported.

1. INTRODUCTION

SIMILARITY solutions for the free convection flow of a Newtonian fluid arising from a steady line thermal source embedded at the leading edge of a vertical adiabatic surface date back to an early paper by Zimin and Lyakhov [1] in 1970.

Later, Jaluria and Gebhart [2] presented accurate numerical solutions of this problem for a Prandtl number range of 0.01–100. Afzal [3] and recently Ingham and Pop [4] have derived higher-order solutions for convective wall plumes for moderately large values of the Grashof number of the method of matched asymptotic expansions. A search of the literature reveals that papers [5–8] are also all devoted to the problem of convective wall plumes in a Newtonian fluid.

The aim of this paper is to investigate the free convection flow of a power-law fluid arising from a line thermal source positioned at the leading edge of a vertical adiabatic surface. A systematic analysis for deriving a possible similarity formulation for this flow problem is presented. In Section 2, the similarity transformation of the boundary layer equations is introduced and numerical solutions are presented for various values of the power-law index, n, and the generalized Prandtl, Pr. The concluding section draws attention to the principal results of this paper.

It is worth mentioning at this point that the transport phenomenon in power-law fluid flow has been the subject of many recent investigations due to the frequent use of this type of fluid in modern industry. Several review articles and books may be consulted for detailed information of this subject [9–14]. To the authors' knowledge no investigation of the problem considered in the present paper has been reported previously in the literature.

2. ANALYSIS

Consider the problem of steady, laminar, free convection from a line source of heat positioned at the leading edge of a vertical adiabatic surface immersed in an unbounded power-law fluid with the following transport properties [15, 16]

$$\tau_{ij} = -P\delta_{ij} + K|0.5J_2|^{(n-1)/2} e_{ij} \tag{1}$$

$$q = -k|0.5J_2|^{s/2} \operatorname{grad} T \tag{2}$$

where τ_{ij} and e_{ij} are the tensors of stress and strain-rate, δ_{ij} is unit tensor, J_2 is the second invariant of the strain-rate tensor, n and s are superscripts identifying non-Newtonian behaviour in the flow and heat transfer. The strict Boussinesq approximation is assumed, i.e. the variation of fluid density with temperature is accounted for only in the buoyancy term of the vertical momentum equation; all other fluid properties are assumed to be constant; and viscous dissipation is neglected. The plume is assumed to be laminar and the boundary layer approximation is assumed to hold. We chose (x^*, y^*) as coordinates with the x^*-axis measured along the wall in an upward direction and the y^*-axis is normal to it. The temperature T takes the value T_∞ in the ambient fluid. Under these conditions the basic equations in non-dimensional form (for details see Table 1) can be written as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3}$$

NOMENCLATURE

C_f	skin friction coefficient
C_p	specific heat at constant pressure
e_{ij}	strain-rate tensor
f	reduced stream function
g	acceleration due to gravity
Gr	generalized Grashof number
h	reduced temperature function
J_2	second invariant of the strain-rate tensor
k	thermal conductivity
K	consistency index
L	reference length
n	flow behaviour index
P	pressure
Pr	generalized Prandtl number
q, I	dimensional and non-dimensional heat input by the thermal source
s	heat transfer behaviour index
T	temperature
T_r	reference temperature
u^*, v^*	velocity components
U	reference velocity
x^*, y^*	Cartesian coordinates.

Greek symbols

α	thermal diffusivity
β	thermal expansion coefficient
δ_{ij}	unit tensor
θ	dimensionless temperature
η	similarity variable
ρ	density
τ	shear stress
ψ	stream function.

Subscripts

w	wall condition
∞	ambient condition.

Superscripts

′	differentiation with respect to η
*	dimensional variables.

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}\left(\left|\frac{\partial u}{\partial y}\right|^{n-1}\frac{\partial u}{\partial y}\right)+\theta \qquad (4)$$

$$u\frac{\partial \theta}{\partial x}+v\frac{\partial \theta}{\partial y}=\frac{1}{Pr}\frac{\partial}{\partial y}\left(\left|\frac{\partial u}{\partial y}\right|^{s}\frac{\partial \theta}{\partial y}\right). \qquad (5)$$

The associated boundary conditions are

$$y=0: u=v=0 \qquad (6a)$$

$$\theta=(T_w-T_\infty)Gr^b/T_r \quad \text{or} \quad \frac{\partial\theta}{\partial y}=0 \qquad (6b)$$

$$y\to\infty: u=0, \quad \theta=0 \qquad (6c)$$

together with a condition which expresses the fact that there is uniform heat flux from the line source

$$\int_0^\infty u\theta\,\mathrm{d}y=I \qquad (7)$$

where

$$I=q(\rho C_p L T_r)^{-1}(\rho L^n Gr^{n-1}/K)^{1/(n-2)}. \qquad (8)$$

The generalized Grashof number is

$$Gr=g\beta T_r L^{(2+n)/(2-n)}/(\rho/K)^{2/(n-2)}$$

and the generalized Prandtl number is

$$Pr=L^{1+s}U^{1-s}\,Gr^{-n(2+s)/(1+4n)}/\alpha$$

where

$$U=(\rho L^n\,Gr^{-n(n+1)/(4n+1)}/K)^{1/(n-2)}$$

is the characteristic velocity for this free convection situation.

The solution of the coupled non-linear partial differential equations (3)–(5) is facilitated by a number of transformations. The first step is to introduce a stream function ψ such that

$$u=\frac{\partial\psi}{\partial y}, \quad v=-\frac{\partial\psi}{\partial x}. \qquad (9)$$

Then, the pseudo-similarity variables are defined as

$$\psi=x^{(2n+1)/(4n+1)}f(x,\eta), \quad \theta=x^{-(2n+1)/(4n+1)}h(x,\eta) \qquad (10a)$$

and

$$\eta=x^{-(n+1)/(4n+1)}y. \qquad (10b)$$

The temperature of the wall is assumed to depend upon x in the following manner:

$$T_w(x)=T_\infty+Gr^{-b}\,T_r x^{-(2n+1)/(4n+1)}. \qquad (11)$$

Insertion of equation (10) into equations (3)–(5) leads to

$$(|f''|^{n-1}f'')'+\frac{2n+1}{4n+1}ff''-\frac{n}{4n+1}f'^2+h=x\left(f'\frac{\partial f'}{\partial x}-f''\frac{\partial f}{\partial x}\right) \qquad (12)$$

Table 1. Dimensionless variables

x	y	u	v	θ
$\frac{x^*}{L}$	$\frac{y^*}{L}Gr^a$	$\frac{u^*}{U}$	$\frac{v^*}{U}Gr^a$	$\frac{T-T_\infty}{T_r}Gr^b$

$a=n/(4n+1)$, $b=(6n-5n-2)/[(4n+1)(n-2)]$

$$x^{(n-1-s)/(4n+1)} \frac{1}{Pr}(|f''|^s h')' + \frac{2n+1}{4n+1}(fh)'$$

$$= x\left(f'\frac{\partial h}{\partial x} - h'\frac{\partial f}{\partial x}\right) \quad (13)$$

subjected to the boundary conditions

$$f(x,0) = f'(x,0) = 0$$

$$h(x,0) = 1 \quad \text{or} \quad h'(x,0) = 0 \quad (14a)$$

$$f'(x,\infty) = h(x,\infty) = 0 \quad (14b)$$

where primes denote partial differentiation with respect to η.

It is apparent that these equations will permit similarity solutions if the exponent of x in equation (13) vanishes, i.e.

$$s = n-1. \quad (15)$$

Under this restrictional condition, equations (12) and (13) become

$$(|f''|^{n-1}f'')' + \frac{2n+1}{4n+1}ff'' - \frac{n}{4n+1}f'^2 + h = 0 \quad (16)$$

$$\frac{1}{Pr}(|f''|^{n-1}h')' + \frac{2n+1}{4n+1}(fh)' = 0 \quad (17)$$

with the boundary conditions of

$$f(0) = f'(0) = 0, \quad h(0) = 1, \quad h'(0) = 0 \quad (18a)$$

$$f'(\infty) = h(\infty) = 0. \quad (18b)$$

The heat flux condition (7) may now be written as

$$I = \int_0^\infty f'h \, d\eta = I(n, Pr). \quad (19)$$

We notice to this end that for $n = 1$, equations (16) and (17) reduce to those of refs. [3, 4] which describe the classical problem of convective wall plume in a Newtonian fluid.

It is customary to present the flow characteristics by means of the skin friction coefficient

$$C_f = 2\frac{\tau_w}{\rho U^2}. \quad (20)$$

Making use of equations (1), (9) and (10) the skin friction coefficient (20) becomes

$$C_f \, Gr^{n/(4n+1)} x^{n/(4n+1)} = 2|f''(0)|^n. \quad (21)$$

In conclusion, the problem to be solved is that presented by equations (16)–(19) and (21).

3. RESULTS AND DISCUSSION

Equations (16) and (17), which are subject to the boundary conditions (18), have been integrated numerically by using the Runge–Kutta–Gill method for n ranging from 0.2 to 2.0 and for $Pr = 10$ and 100, respectively. As mentioned in Section 2, in this model n is the property of a fluid with $n = 1$ for a Newtonian fluid. Non-Newtonian fluids with $n < 1$ are called pseudo-plastic (most macromolecular fluids are of this kind with $0.2 < n < 0.6$, see Bird *et al.* [17]) and those with $n > 1$ dilatant.

Table 2. Numerical values of computed parameters for various n and $Pr = 10$

n	$f''(0)$	I	f'_{max}	$f(\infty)$	$\eta(f'_{max})$	$\eta_{0.50}$
0.2	3.21309	14.67177	0.70945	3.46476	1.8000	3.25674
0.4	1.56230	9.05100	0.56938	2.48399	1.6500	2.15474
0.6	1.08804	5.18968	0.45349	1.52345	1.3000	1.49193
0.8	0.95063	3.96055	0.43117	1.32087	1.2000	1.24216
1.0	0.86123	3.00209	0.39276	0.99482	1.0000	1.04663
1.2	0.90450	2.87300	0.49054	1.62756	1.4000	0.92290
1.5	0.82903	2.12471	0.40676	0.70861	0.9000	0.82179
2.0	0.76044	1.27685	0.30440	0.32992	0.6000	0.64404

The results for various transport parameters, which are important for representing the heat transfer correlations, see Gebhart *et al.* [18], are given in Tables 2 and 3 for the flow behaviour index ranging from 0.2 to 2.0 and Prandtl numbers of 10 and 100, respectively. In order to assess the accuracy of our numerical results, the present results were compared with those of ref. [4] for $n = 1$ (Newtonian fluids). Thus, the values for $C_f Gr^{1/5} x^{1/5}$ from equation (21) when $n = 1$ are 2.62012 for $Pr = 0.72$ and 1.85964 for $Pr = 6.7$ while the corresponding values from ref. [4] are 2.6201 and 1.8596, respectively. This shows that the agreement is excellent. Also, comparison with the results of Liburdy and Faeth [19] for $n = 1$ with $Pr = 10$ and 100 is found to be very good.

It is noted that from the present results that the friction factor decreases with increasing values of n and Pr. This fact is also verified from the results presented in Fig. 1. The integral I defined in equation (19) determines the velocity level and the surface temperature. From Table 3 we observe that I decreases with increasing values of Pr and n. Note that having determined the values of I, the reference temperature T_r can be obtained easily through equation (8).

Figures 2 and 3 display results for the upward velocity profiles in the wall plume. It is observed that the maximum velocity decreases with increasing values of the flow behaviour index, n. The location of the maximum velocity moves closer to the wall as n

Table 3. Numerical values of computed parameters for various n and $Pr = 100$

n	$f''(0)$	I	f'_{max}	$f(\infty)$	$\eta(f'_{max})$	$\eta_{0.50}$
0.2	1.23283	3.33278	0.22884	0.90717	1.5000	2.19335
0.4	0.73858	2.33788	0.20504	0.73486	1.3000	0.87350
0.6	0.50828	1.04260	0.13315	0.19407	0.8000	1.02645
0.8	0.41895	0.53576	0.09548	0.08289	0.5400	0.76715
1.0	0.48062	0.52386	0.12145	0.12585	0.6000	0.59084
1.2	0.54364	0.48404	0.16156	0.19320	0.8000	0.49105
1.5	0.53094	0.31474	0.13440	0.12537	0.5000	0.41985
2.0	0.44790	0.10934	0.06097	0.01668	0.2100	0.25987

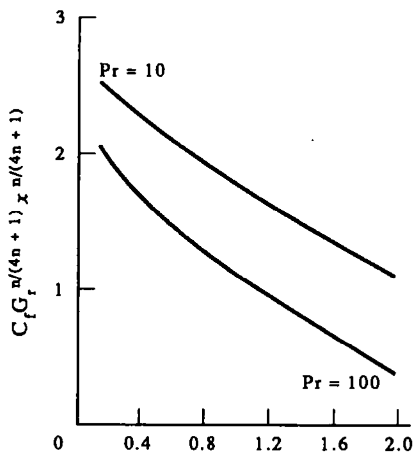

FIG. 1. Friction factor vs flow behaviour index, n, for $Pr = 10$ and 100.

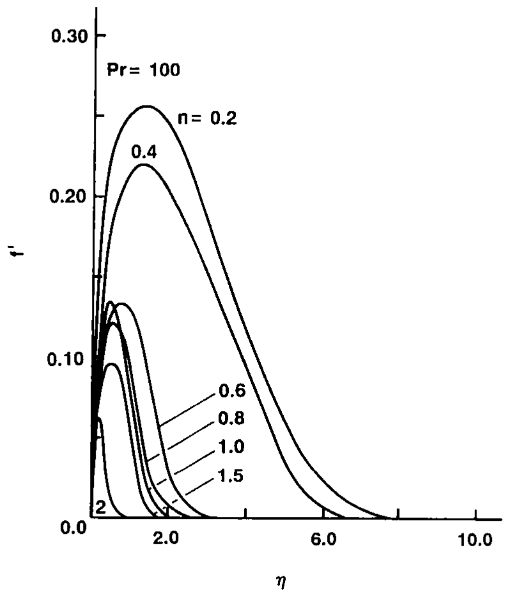

FIG. 3. Upward velocity profile vs similarity variable, η, for $Pr = 100$.

increases. The boundary region thickness decreases as n increases. As the Prandtl number increases, the thinning effect on the thermal layer influences the boundary region. Also, it is remarkable that the curve for $n = 1$ (Newtonian fluids) appears to intersect more curves for $Pr = 100$ than for $Pr = 10$. The reason for this seems to be the dependence of the Prandtl number on the flow behaviour index, n, reference velocity, U, and the reference length, L, of the vertical surface. The results from Figs. 4 and 5 describe the temperature distribution in the wall plume. It is observed that as Pr or n increases, the thermal layer becomes thinner. The important considerations in this flow are the velocity level, the surface temperature and the extent of the boundary region. As the flow proceeds downstream from a heated element located on an unheated vertical surface, it influences the cooling characteristics of any other elements it may encounter. An element downstream is immersed in a flowing heated fluid, whose temperature and velocity are determined by the distance between the two elements and the heat flux input I. Tables 2 and 3 show the necessary values of $f''(0)$, I, f'_{max}, $f(\infty)$ and $\eta(f'_{max})$, over the Pr and n ranges considered, to allow evaluation of the temperature and velocity fields at a downstream element.

The flow induced by buoyancy due to a horizontal

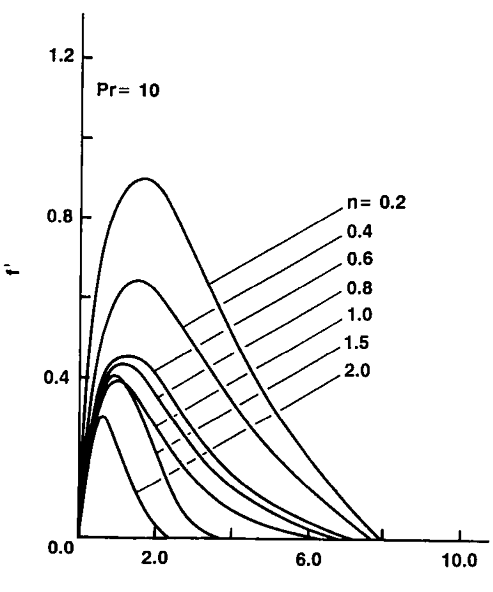

FIG. 2. Upward velocity profile vs similarity variable, η, for $Pr = 10$.

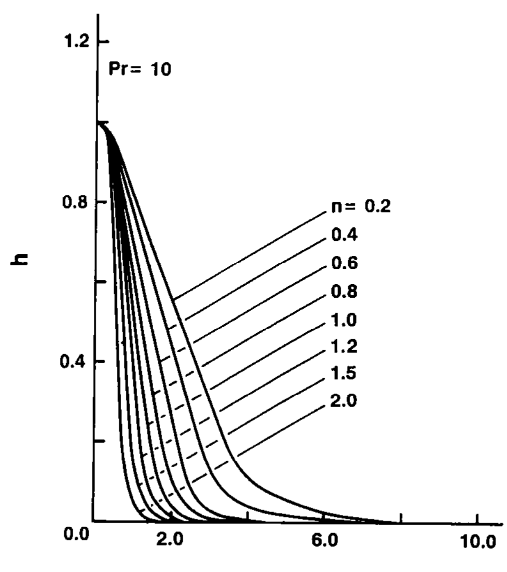

FIG. 4. Temperature distribution in the wall plume vs similarity variable, η, for $Pr = 10$.

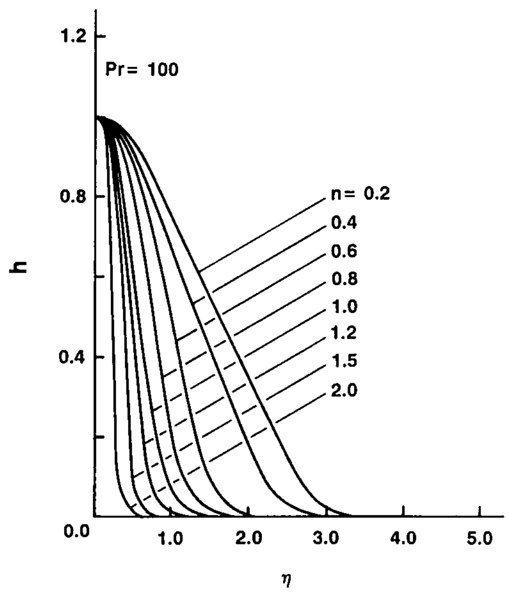

FIG. 5. Temperature distribution in the wall plume vs similarity variable, η, for $Pr = 100$.

line source embedded at the leading edge of a vertical plate in a power-law fluid is an interesting practical problem. It is hoped that the present work will elicit some experiments for quick and yet accurate estimations of convective heat transfer rates.

4. CONCLUDING REMARKS

In this paper, we have analysed the laminar natural convection flow generated by a line thermal source imbedded in an adiabatic vertical surface. The flow configuration is of much interest since the governing boundary layer equations admit similarity solutions which are more revealing than the direct numerical integration of the partial differential equations. In addition, this problem is of considerable importance in engineering applications, such as the positioning of components dissipating energy on vertical circuit boards, and the results concerning the boundary layer flow characteristics are reported here. The numerical results presented in this paper allow evaluation of the velocity and temperature fields in the generated flow. The flow behaviour index was varied from 0.2 to 2.0 whereas the Prandtl number was taken as 10 and 100.

Acknowledgement—The authors greatly appreciate the constructive suggestions that were made by the reviewers of the original manuscript.

REFERENCES

1. V. D. Zimin and Y. N. Lyakhov, Convective wall plume, *J. Appl. Mech. Tech. Phys.* **11**, 159–161 (1970).
2. Y. Jaluria and B. Gebhart, Buoyancy-induced flow arising from a line thermal source on an adiabatic vertical surface, *Int. J. Heat Mass Transfer* **20**, 153–157 (1977).
3. N. Afzal, Convective wall plume: higher order analysis, *Int. J. Heat Mass Transfer* **23**, 505–513 (1980).
4. D. B. Ingham and I. Pop, A note on the free convection in a wall plume: horizontal wall effects, *Int. J. Heat Mass Transfer* **33**, 1770–1773 (1990).
5. K. V. Rao, B. F. Armaly and T. S. Chen, Analysis of laminar mixed convective plumes along vertical adiabatic surfaces, *J. Heat Transfer* **106**, 552–557 (1984).
6. R. Krishnamurthy and B. Gebhart, Mixed convection in wall plumes, *Int. J. Heat Mass Transfer* **27**, 1679–1689 (1984).
7. H.-T. Lin and J.-J. Chen, Mixed convection wall plumes, *Int. J. Heat Mass Transfer* **30**, 1721–1726 (1987).
8. Y. Joshi, Wall plume at extreme Prandtl numbers, *Int. J. Heat Mass Transfer* **30**, 2686–2690 (1987).
9. Z. P. Shulman and B. M. Berkovskiy, *Boundary Layer Theory of Non-Newtonian Fluids*. Nauka i Technika, Minsk (1966) (in Russian).
10. Z. P. Shulman, *Heat and Mass Exchange in Non-Newtonian Fluids*. Energya, Moscow (1975) (in Russian).
11. Z. P. Shulman, B. I. Baykov and E. A. Zaltsgendler, *Heat and Mass Transfer in Free Convection of Non-Newtonian Fluids*. Nauka i Technika, Minsk (1975) (in Russian).
12. Y. I. Cho and J. P. Hartnett, Non-Newtonian fluids in circular pipe flow, *Adv. Heat Transfer* **15**, 59–141 (1982).
13. A. V. Shenoy and R. A. Mashelkar, Thermal convection in non-Newtonian fluids, *Adv. Heat Transfer* **15**, 143–225 (1982).
14. T. F. Irvine, Jr. and J. Karni, Non-Newtonian fluid flow and heat transfer. In *Handbook of Single-phase Convective Heat Transfer* (Edited by S. Kakac *et al.*), pp. 20.1–20.57. Wiley, New York (1987).
15. Yu. I. Shvets and V. K. Vishnevskiy, Effect of dissipation on convective heat transfer in flow of non-Newtonian fluids, *Heat Transfer—Sov. Res.* **19**, 38–43 (1987).
16. S. Nakamura, R. S. R. Gorla and I. Pop, Free convection similarity solution of a non-Newtonian fluid on a horizontal surface, *Int. J. Engng Fluid Mech.* (in press).
17. R. B. Bird, R. C. Armstrong and O. Hassager, *Dynamics of Polymer Liquids*. Wiley, New York (1977).
18. B. G. Gebhart, Y. Jaluria, R. L. Mahajan and B. Sammakia, *Buoyancy-induced Flows and Transport*. Hemisphere, New York (1988).
19. J. A. Liburdy and G. M. Faeth, Theory of a steady laminar thermal plume along a vertical adiabatic wall, *Lett. Heat Mass Transfer* **2**, 407–418 (1975).